

# Division Theorems for the Koszul Complex

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## Abstract

We establish a twisted version of Skoda's estimate for the Koszul complex from which we get global division theorems for the Koszul complex. This generalizes Skoda's division theorem. We also show how to use Skoda triples to produce division theorems for the Koszul complex.

## 1 Introduction

Skoda's division theorem is a  $L^2$  version of the Corona theorem in higher dimensions. It turns out to be an important tool in establishing effective results in complex geometry and algebraic geometry(see refs.[B87],[El99],[Siu98] and [Siu00]).

Many generalizations of Skoda's division have been known since [S72]. In [S78] and [D82], division theorems were proved for generically surjective homomorphisms between holomorphic vector bundles. Inspired by the Ohsawa-Takegoshi technique, Varolin ([V08]) proved the twisted version of Skoda's estimate and also introduced Skoda triple which enabled him to get a series of Skoda-type theorems. By using the method of residue currents, Andersson also studied division problem for the Koszul complex and its geometric applications(see [A04],[AG11] and references therein).

In this paper, we first prove a Skoda-type estimate(see lemma 2) for the Koszul complex and then try to introduce twisting into such an a priori estimate. To prove lemma 1, we make use of a generalization of Skoda's inequality whose proof is included in the appendix. Based upon the twisted version of our Skoda-type estimate(see lemma 3), we obtain Skoda-type division theorems for the Koszul complex. In principal, by using the Skoda triple introduced by Varolin, we can obtain many examples of division theorems for the Koszul complex. Moreover, the technique of denominators ([MV07]) could be used to produce Skoda triples as shown in [V08]. Our main results on the division problem are theorems 1 and 2. As an application of theorem 1, we give a sufficient condition(see corollary 1) under which the Koszul complex induces an exact sequence at the level of global sections. We give explicit examples(corollaries 2 and 3) from theorem 2 and choosing Skoda triples, then we also discuss the relations among these results.

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## 2 Twisted Estimate for the Koszul Complex

Let  $\Omega$  be a domain in  $\mathbb{C}^n$ ,  $g_1, \dots, g_p \in \mathcal{O}(\Omega)$ , we denote by  $g$  the vector-valued function  $(g_1, \dots, g_p)$  where  $\mathcal{O}(\Omega)$  is the ring of holomorphic functions on  $\Omega$ . We also denote by  $\mathcal{O}_\Omega$  the sheaf of germs of holomorphic functions on  $\Omega$ . Now we can introduce the well-known Koszul complex

$$0 \rightarrow \bigwedge^p \mathcal{O}_\Omega^{\oplus p} \xrightarrow{\iota_g} \dots \xrightarrow{\iota_g} \bigwedge^\ell \mathcal{O}_\Omega^{\oplus p} \xrightarrow{\iota_g} \bigwedge^{\ell-1} \mathcal{O}_\Omega^{\oplus p} \xrightarrow{\iota_g} \dots \xrightarrow{\iota_g} \mathcal{O}_\Omega \rightarrow 0 \quad (1)$$

The sheaf-homomorphism  $\iota_g$  is defined for each  $1 \leq \ell \leq p$  as follows

$$(\iota_g v)_{i_1 \dots i_{\ell-1}} = \sum_{1 \leq i \leq p} g_i v_{i i_1 \dots i_{\ell-1}}, \quad 1 \leq i_1, \dots, i_{\ell-1} \leq p \quad (2)$$

where  $v = (v_{i_1 \dots i_\ell})_{i_1 \dots i_\ell=1}^p \in \bigwedge^\ell \mathcal{O}_\Omega^{\oplus p}$  i.e.  $v_{i_1 \dots i_\ell} \in \mathcal{O}_\Omega$  and  $v_{i_1 \dots i_\ell}$  are skew-symmetric on the indices  $i_1, \dots, i_\ell$ .

We consider, in this paper, the global division problem for the Koszul complex, i.e. to find sufficient condition of integrability under which an element  $f \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p}$  ( $1 \leq \ell \leq p$ ) with  $\iota_g f = 0$  should be contained in the image of  $\iota_g$ , that is, there exists some  $u \in \bigwedge^\ell \mathcal{O}(\Omega)^{\oplus p}$  such that  $f = \iota_g u$ . The celebrated Skoda division theorem is concerned with the case where  $\ell = 1$  of this problem.

We will agree on the following index ranges:

$$1 \leq i, j, k, \ell \leq p, \quad 1 \leq \alpha, \beta \leq n.$$

In the remaining part of this section, we assume that  $\Omega$  is a bounded domain with smooth boundary and  $g_i \in \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega})$ ,  $1 \leq i \leq p$ , which have no common zeros on  $\bar{\Omega}$ . Such assumptions will be dropped in section 2 by standard argument. For functions  $\phi_1, \phi_2 \in C^2(\bar{\Omega})$ , we define

$$\iota_g^* h = e^{\phi_1 - \phi_2} \bar{g} \wedge h \quad (3)$$

where  $h \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p}$ ,  $1 \leq \ell \leq p$  and  $\bar{g} \wedge h$  is given by

$$(\bar{g} \wedge h)_{i_1 \dots i_\ell} = - \sum_{1 \leq \sigma \leq \ell} (-1)^\sigma \bar{g}_{i_\sigma} h_{i_1 \dots \hat{i}_\sigma \dots i_\ell}. \quad (4)$$

where  $\hat{i}_\sigma$  means that the index  $i_\sigma$  is omitted. To formulate our a priori estimate, we need to introduce the following space.

$$F = \{h \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p} \mid \iota_g h = 0\}. \quad (5)$$

Now we start to estimate  $\|\iota_g^* h + \bar{\partial}_{\phi_1}^* v\|_{\phi_1}^2$  from below for an arbitrary  $h \in F$  which is smooth on  $\bar{\Omega}$  and  $v \in \text{Dom} \bar{\partial}_{\phi_1}^* \subseteq \bigwedge^\ell L_{0,1}^2(\Omega, \phi_1)^{\oplus p}$  satisfying  $\bar{\partial} v = 0$ . The reason

why we should estimate such a term lies in Skoda's fundamental lemma (see lemma 4 at the end of this section).

$$\begin{aligned}\|\iota_g^* h + \bar{\partial}_{\phi_1}^* v\|_{\phi_1}^2 &= \|\iota_g^* h\|_{\phi_1}^2 + 2\operatorname{Re}(\iota_g^* h, \bar{\partial}_{\phi_1}^* v)_{\phi_1} + \|\bar{\partial}_{\phi_1}^* v\|_{\phi_1}^2 \\ &= I + II + III.\end{aligned}$$

By definition, we have

$$I = \int_{\Omega} \|\bar{g} \wedge h\|^2 e^{\phi_1 - 2\phi_2} dV.$$

Form the following identity

$$\begin{aligned}\|\bar{g} \wedge h\|^2 &= \frac{1}{\ell!} \sum_{\substack{1 \leq \sigma, \delta \leq \ell \\ 1 \leq i_1, \dots, i_\ell \leq p}} (-1)^{\sigma+\delta} \bar{g}_{i_\sigma} g_{i_\delta} h_{i_1 \dots \hat{i}_\sigma \dots i_\ell} \bar{h}_{i_1 \dots \hat{i}_\delta \dots i_\ell} \\ &= \frac{1}{\ell!} \sum_{\substack{1 \leq \sigma \leq \ell \\ 1 \leq i_1, \dots, i_\ell \leq p}} |g_{i_\sigma}|^2 |h_{i_1 \dots \hat{i}_\sigma \dots i_\ell}|^2 \\ &\quad - \frac{1}{\ell!} \sum_{\substack{1 \leq \sigma \neq \delta \leq \ell \\ 1 \leq i_1, \dots, \hat{i}_\sigma, \dots, \hat{i}_\delta, \dots, i_\ell \leq p}} \sum_{1 \leq i_\sigma \leq p} \bar{g}_{i_\sigma} \bar{h}_{i_\sigma i_1 \dots \hat{i}_\sigma \dots \hat{i}_\delta \dots i_\ell} \sum_{1 \leq i_\delta \leq p} g_{i_\delta} h_{i_\delta i_1 \dots \hat{i}_\sigma \dots \hat{i}_\delta \dots i_\ell} \\ &\stackrel{\iota_g h=0}{=} \frac{1}{(\ell-1)!} \sum_{1 \leq i, i_1, \dots, i_{\ell-1} \leq p} |g_i|^2 |h_{i_1 \dots i_{\ell-1}}|^2 \\ &= \|g\|^2 \|h\|^2,\end{aligned}$$

it follows that

$$I = \int_{\Omega} \|g\|^2 \|h\|^2 e^{\phi_1 - 2\phi_2} dV$$

where  $\|g\|^2 = \sum_{1 \leq i \leq p} |g_i|^2$ ,  $\|h\|^2 = \frac{1}{(\ell-1)!} \sum_{1 \leq i_1, \dots, i_{\ell-1} \leq p} |h_{i_1 \dots i_{\ell-1}}|^2$ .

Assuming that  $\phi_1, \phi_2$  are related as follows

$$\phi_2 = \phi_1 + \log \|g\|^2, \tag{6}$$

then we have

$$I = \|h\|_{\phi_2}^2. \tag{7}$$

Moreover, with  $\phi_1, \phi_2$  chosen in such a way, we also have

$$\iota_g^* = \frac{\bar{g}}{\|g\|^2} \wedge.$$

To deal with the term  $II$ , we first use integration by parts to move the operator  $\bar{\partial}_{\phi_1}$  from right to left.

$$\begin{aligned}
2\operatorname{Re}(\iota_g^* h, \bar{\partial}_{\phi_1}^* v)_{\phi_1} &= 2\operatorname{Re} \int_{\Omega} \left( \frac{\bar{g}}{\|g\|^2} \wedge h, \bar{\partial}_{\phi_1}^* v \right) e^{-\phi_1} dV \\
&= 2\operatorname{Re} \int_{\Omega} \left( \bar{\partial} \left( \frac{\bar{g}}{\|g\|^2} \wedge h \right), v \right) e^{-\phi_1} dV \\
&= 2\operatorname{Re} \int_{\Omega} \left( \sum_{1 \leq \alpha \leq n} \overline{\partial_{\alpha} \left( \frac{g}{\|g\|^2} \right)} \wedge h d\bar{z}_{\alpha}, v \right) e^{-\phi_1} dV.
\end{aligned}$$

Let

$$v_{i_1 \dots i_{\ell}} = \sum_{1 \leq \alpha \leq n} v_{i_1 \dots i_{\ell} \alpha} d\bar{z}_{\alpha},$$

then we have

$$\begin{aligned}
II &= \frac{-2\operatorname{Re}}{\ell!} \int_{\Omega} \sum_{\substack{1 \leq \sigma \leq \ell \\ 1 \leq \alpha \leq n \\ 1 \leq i_1, \dots, i_{\ell} \leq p}} (-1)^{\sigma} \overline{\partial_{\alpha} (\|g\|^{-2} g_{i_{\sigma}})} h_{i_1 \dots i_{\sigma} \dots i_{\ell}} \overline{v_{i_1 \dots i_{\ell} \alpha}} e^{-\phi_1} dV \\
&= \frac{2\operatorname{Re}}{(\ell-1)!} \int_{\Omega} \sum_{\substack{1 \leq \alpha \leq n \\ 1 \leq i, i_1, \dots, i_{\ell-1} \leq p}} h_{i_1 \dots i_{\ell-1}} \overline{\partial_{\alpha} (\|g\|^{-2} g_i) v_{ii_1 \dots i_{\ell-1} \alpha}} e^{-\phi_1} dV
\end{aligned}$$

which gives

$$\begin{aligned}
II &\geq - \int_{\Omega} \frac{1}{b} \|h\|^2 e^{-\phi_2} dV \\
&\quad - \frac{1}{(\ell-1)!} \sum_{1 \leq i_1, \dots, i_{\ell-1} \leq p} \int_{\Omega} b \|g\|^2 \left| \sum_{\substack{1 \leq \alpha \leq n \\ 1 \leq i \leq p}} \partial_{\alpha} (\|g\|^{-2} g_i) v_{ii_1 \dots i_{\ell-1} \alpha} \right|^2 e^{-\phi_1} dV
\end{aligned}$$

where  $b > 1$  is an arbitrary measurable function on  $\Omega$ .

To handle the second term in the above inequality, we need the following lemma which is a generalization of Skoda's inequality ([S72], page 552).

**Lemma 1** Given constants  $a_i, b_{i\alpha}, c_{i_1 \dots i_{\ell} \alpha} \in \mathbb{C} (1 \leq i, \ell, i_1, \dots, i_{\ell} \leq p, 1 \leq \alpha \leq n, p, n \in \mathbb{N})$  where  $c_{i_1 \dots i_{\ell} \alpha}$  are skew-symmetric with respect to  $i_1 \dots i_{\ell}$ . Then for any  $1 \leq i_1 < \dots < i_{\ell-1} \leq p$ , it holds that

$$\left| \sum_{\substack{1 \leq i, j \leq p \\ 1 \leq \alpha \leq n}} \overline{a_j} (a_j b_{i\alpha} - a_i b_{j\alpha}) c_{ii_1 \dots i_{\ell-1} \alpha} \right|^2 \leq q \sum_{1 \leq i \leq p} |a_i|^2 \sum_{\substack{1 \leq i \leq p \\ 1 \leq j < k \leq p}} \left| \sum_{1 \leq \alpha \leq n} (a_j b_{k\alpha} - a_k b_{j\alpha}) c_{ii_1 \dots i_{\ell-1} \alpha} \right|^2 \quad (8)$$

where  $q$  is the positive constant defined by

$$q = \begin{cases} \min\{p-1, n\}, & \ell = 1; \\ \min\{p-\ell+1, n\}, & \ell \geq 2. \end{cases}$$

We postpone our proof of lemma 1 to the appendix. Now we continue the estimate for  $II$ . Applying lemma 1 with

$$a_i = g_i, \quad b_{i\alpha} = \partial_\alpha g_i, \quad c_{i_1 \dots i_\ell} = v_{i_1 \dots i_\ell},$$

we obtain for fixed  $1 \leq i_1 < \dots < i_{\ell-1} \leq p$  that

$$\begin{aligned} \|g\|^2 \left| \sum_{\substack{1 \leq \alpha \leq n \\ 1 \leq i \leq p}} \partial_\alpha (\|g\|^{-2} g_i) v_{ii_1 \dots i_{\ell-1} \alpha} \right|^2 &= \|g\|^{-6} \left| \sum_{\substack{1 \leq \alpha \leq n \\ 1 \leq i, j \leq p}} \bar{g}_j (g_j \partial_\alpha g_i - g_i \partial_\alpha g_j) v_{ii_1 \dots i_{\ell-1} \alpha} \right|^2 \\ &\leq q \|g\|^{-4} \sum_{\substack{1 \leq i \leq p \\ 1 \leq j < k \leq p}} \left| \sum_{1 \leq \alpha \leq n} (g_j \partial_\alpha g_k - g_k \partial_\alpha g_j) v_{ii_1 \dots i_{\ell-1} \alpha} \right|^2 \\ &\stackrel{(*)}{=} q \|g\|^{-2} \sum_{1 \leq j, k \leq p} \left| \sum_{1 \leq \alpha \leq n} \partial_\alpha g_k v_{ji_1 \dots i_{\ell-1} \alpha} \right|^2 \\ &\quad - q \|g\|^{-4} \sum_{1 \leq j \leq p} \left| \sum_{\substack{1 \leq k \leq p \\ 1 \leq \alpha \leq n}} g_k \partial_\alpha g_k v_{ji_1 \dots i_{\ell-1} \alpha} \right|^2 \\ &= q \sum_{\substack{1 \leq j \leq p \\ 1 \leq \alpha, \beta \leq n}} \partial_\alpha \partial_\beta \log \|g\|^2 \cdot v_{ji_1 \dots i_{\ell-1} \alpha} \overline{v_{ji_1 \dots i_{\ell-1} \beta}} \end{aligned}$$

where  $q$  is the constant in lemma 1. We have used the Lagrange identity to get (\*).

Consequently, we can estimate  $II$  from below as follows.

$$\begin{aligned} II &\geq - \int_{\Omega} \frac{1}{b} \|h\|^2 e^{-\phi_2} dV \\ &\quad - \frac{q}{(\ell-1)!} \sum_{\substack{1 \leq i_1, \dots, i_\ell \leq p \\ 1 \leq \alpha, \beta \leq n}} \int_{\Omega} b \partial_\alpha \partial_\beta \log \|g\|^2 \cdot v_{i_1 \dots i_\ell \alpha} \overline{v_{i_1 \dots i_\ell \beta}} e^{-\phi_1} dV. \end{aligned} \quad (9)$$

Since  $I$  only involves  $h$ , we want to control the second term in (9) by  $III$ . By using the standard Kohn-Morrey-Hörmander identity and the boundary condition

$v \in \text{Dom} \bar{\partial}_{\phi_1}^*$ , we can estimate, in the case where  $\Omega$  is assumed additionally to be pseudoconvex, the last term  $III$  as follows

$$\begin{aligned} III &= \frac{1}{\ell!} \sum_{1 \leq i_1, \dots, i_\ell \leq p} \int_{\Omega} |\bar{\partial}_{\phi_1}^* v_{i_1 \dots i_\ell}|^2 e^{-\phi_1} dV \\ &\geq \frac{1}{\ell!} \sum_{\substack{1 \leq i_1, \dots, i_\ell \leq p \\ 1 \leq \alpha, \beta \leq n}} \int_{\Omega} \partial_{\alpha} \bar{\partial}_{\beta} \phi_1 v_{i_1 \dots i_\ell \alpha} \overline{v_{i_1 \dots i_\ell \beta}} e^{-\phi_1} dV. \end{aligned} \quad (10)$$

for any  $v \in \text{Dom} \bar{\partial}_{\phi_1}^* \subseteq \bigwedge^{\ell} L_{0,1}^2(\Omega, \phi_1)^{\oplus p}$  satisfying  $\bar{\partial}v = 0$ .

Taking the sum of (7), (9) and (10), we get the desired Skoda-type estimate for the Koszul complex.

**Lemma 2** Let  $\Omega$  be a bounded pseudoconvex domain with smooth boundary and  $g_i \in \mathcal{O}(\Omega) \cap C^{\infty}(\bar{\Omega})$  ( $1 \leq i \leq p$ ) which have no common zeros on  $\bar{\Omega}$ . We assume that  $\phi_1, \phi_2 \in C^2(\bar{\Omega})$  are functions satisfying (6), and  $b > 1$  is a measurable function on  $\Omega$ . Then for any  $h \in F$  and any  $v \in \text{Dom} \bar{\partial}_{\phi_1}^* \subseteq \bigwedge^{\ell} L_{0,1}^2(\Omega, \phi_1)^{\oplus p}$  satisfying  $\bar{\partial}v = 0$ , it holds that

$$\begin{aligned} \|\iota_g^* h + \bar{\partial}_{\phi_1}^* v\|_{\phi_1}^2 &\geq \int_{\Omega} \frac{b-1}{b} \|h\|^2 e^{-\phi_2} dV \\ &\quad + \frac{1}{\ell!} \sum_{\substack{1 \leq i_1, \dots, i_\ell \leq p \\ 1 \leq \alpha, \beta \leq n}} \int_{\Omega} (\partial_{\alpha} \bar{\partial}_{\beta} \phi_1 - q \ell b \partial_{\alpha} \bar{\partial}_{\beta} \log \|g\|^2) v_{i_1 \dots i_\ell \alpha} \overline{v_{i_1 \dots i_\ell \beta}} e^{-\phi_1} dV. \end{aligned} \quad (11)$$

Now we want to introduce twisting into the apriori estimate (11). Following [V08], we twist simultaneously the weights  $\phi_1, \phi_2$  by a function  $0 < a \in C^2(\bar{\Omega})$  and consider the following new weights.

$$\varphi_1 = \phi_1 + \log a, \quad \varphi_2 = \phi_2 + \log a. \quad (12)$$

From (6) and (12), it follows that

$$\varphi_2 = \varphi_1 + \log \|g\|^2. \quad (13)$$

By the definition (3) of  $\iota_g^*$ , we know

$$\iota_g^* = e^{\varphi_1 - \varphi_2} \bar{g} \wedge. \quad (14)$$

From the definition of  $\bar{\partial}_{\phi_1}^*$  and (12), we get

$$(\bar{\partial}_{\phi_1}^* v)_{i_1 \dots i_\ell} = (\bar{\partial}_{\varphi_1}^* v)_{i_1 \dots i_\ell} - \sum_{1 \leq \alpha \leq n} \frac{\partial_{\alpha} a}{a} v_{i_1 \dots i_\ell \alpha}.$$

which implies the following identity

$$\begin{aligned}
\|\sqrt{a}\iota_g^*h + \sqrt{a}\bar{\partial}_{\varphi_1}^*v\|_{\varphi_1}^2 &= \|\iota_g^*h + \bar{\partial}_{\varphi_1}^*v\|_{\phi_1}^2 \\
&= \|\iota_g^*h + \bar{\partial}_{\phi_1}^*v\|_{\phi_1}^2 \\
&\quad + \frac{1}{\ell!} \sum_{\substack{1 \leq \alpha, \beta \leq 1 \\ 1 \leq i_1, \dots, i_\ell \leq p}} \int_{\Omega} a^{-2} \partial_{\alpha} a \partial_{\bar{\beta}} a v_{i_1 \dots i_{\ell} \alpha} \overline{v_{i_1 \dots i_{\ell} \beta}} e^{-\phi_1} dV \\
&\quad + \frac{2}{\ell!} \operatorname{Re} \sum_{1 \leq i_1, \dots, i_{\ell} \leq p} \int_{\Omega} \frac{1}{a} (\iota_g^*h + \bar{\partial}_{\phi_1}^*v)_{i_1 \dots i_{\ell}} \overline{\sum_{1 \leq \alpha \leq n} \partial_{\alpha} a v_{i_1 \dots i_{\ell} \alpha} e^{-\phi_1}} dV \\
&= \|\iota_g^*h + \bar{\partial}_{\phi_1}^*v\|_{\phi_1}^2 \\
&\quad - \frac{1}{\ell!} \sum_{\substack{1 \leq \alpha, \beta \leq 1 \\ 1 \leq i_1, \dots, i_{\ell} \leq p}} \int_{\Omega} a^{-2} \partial_{\alpha} a \partial_{\bar{\beta}} a v_{i_1 \dots i_{\ell} \alpha} \overline{v_{i_1 \dots i_{\ell} \beta}} e^{-\phi_1} dV \\
&\quad + \frac{2}{\ell!} \operatorname{Re} \sum_{1 \leq i_1, \dots, i_{\ell} \leq p} \int_{\Omega} \frac{1}{a} (\iota_g^*h + \bar{\partial}_{\phi_1}^*v)_{i_1 \dots i_{\ell}} \overline{\sum_{1 \leq \alpha \leq n} \partial_{\alpha} a v_{i_1 \dots i_{\ell} \alpha} e^{-\phi_1}} dV.
\end{aligned}$$

Substituting (11) and the following equation

$$\partial_{\alpha} \partial_{\bar{\beta}} \phi_1 = \partial_{\alpha} \partial_{\bar{\beta}} \varphi_1 - a^{-1} \partial_{\alpha} \partial_{\bar{\beta}} a + a^{-2} \partial_{\alpha} a \partial_{\bar{\beta}} a$$

into the above identity, it follows that

$$\begin{aligned}
\|\sqrt{a}\iota_g^*h + \sqrt{a}\bar{\partial}_{\varphi_1}^*v\|_{\varphi_1}^2 &\geq \int_{\Omega} \frac{b-1}{b} \|h\|^2 e^{-\phi_2} dV \\
&\quad + \frac{1}{\ell!} \sum_{\substack{1 \leq \alpha, \beta \leq 1 \\ 1 \leq i_1, \dots, i_{\ell} \leq p}} \int_{\Omega} [\partial_{\alpha} \partial_{\bar{\beta}} \varphi_1 - q\ell b \partial_{\alpha} \partial_{\bar{\beta}} \log \|g\|^2 \\
&\quad - \frac{1}{a} \partial_{\alpha} \partial_{\bar{\beta}} a - \frac{1}{a\lambda} \partial_{\alpha} a \partial_{\bar{\beta}} a] v_{i_1 \dots i_{\ell} \alpha} \overline{v_{i_1 \dots i_{\ell} \beta}} e^{-\phi_1} dV \\
&\quad - \|\sqrt{\lambda} \iota_g^*h + \sqrt{\lambda} \bar{\partial}_{\varphi_1}^*v\|_{\varphi_1}^2.
\end{aligned}$$

where  $\lambda > 0$  is a measurable function on  $\Omega$ .

If the following condition holds

$$a \partial_{\alpha} \partial_{\bar{\beta}} \phi_1 - \partial_{\alpha} \partial_{\bar{\beta}} a - \lambda^{-1} \partial_{\alpha} a \partial_{\bar{\beta}} a \geq q\ell ab \partial_{\alpha} \partial_{\bar{\beta}} \log \|g\|^2 \quad (15)$$

where both sides are understood as symmetric sesquilinear forms and  $q$  is the constant in lemma 1, then we have

$$\|\sqrt{a + \lambda} \iota_g^*h + \sqrt{a + \lambda} \bar{\partial}_{\varphi_1}^*v\|_{\varphi_1}^2 \geq \int_{\Omega} \frac{(b-1)a}{b} \|h\|^2 e^{-\varphi_2} dV. \quad (16)$$

where  $\iota_g^*$  is given by (14). The estimate (16) is a twisted version of (11).

We summarize previously obtained estimates in the following lemma.

**Lemma 3** Let  $\Omega$  be a bounded pseudoconvex domain with smooth boundary and  $g_i \in \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega})$  ( $1 \leq i \leq p$ ) which have no common zeros on  $(\bar{\Omega})$ . We assume that  $\varphi_1, \varphi_2 \in C^2(\bar{\Omega})$  are functions satisfying (13),  $0 < a \in C^2(\bar{\Omega})$  and  $1 < b, 0 < \lambda$  are measurable functions on  $\Omega$ . Then for any  $h \in F$  and any  $v \in \text{Dom} \bar{\partial}_{\varphi_1}^* \subseteq \bigwedge^\ell L_{0,1}^2(\Omega, \varphi_1)^{\oplus p}$  satisfying  $\bar{\partial}v = 0$ , the twisted estimate (16) follows from the condition (15).

The next fundamental lemma reduces the problem of establishing division theorems to an a priori estimate (see [S72] and [V08] for proofs).

**Lemma 4** Let  $H, H_0, H_1, H_2$  be Hilbert spaces,  $T : H_0 \rightarrow H$  be a bounded linear operator,  $T_\nu : H_{\nu-1} \rightarrow H_\nu$  ( $\nu = 1, 2$ ) be linear, closed, densely defined operators such that  $T_2 \circ T_1 = 0$ , and let  $F \subseteq H$  be a closed subspace such that  $T(\text{Ker} T_1) \subseteq F$ . Then for every  $f \in F$  and constant  $C > 0$  the following statements are equivalent

1. There exists at least one  $u \in \text{Ker} T_1$  such that  $Tu = f$ ,  $\|u\|_{H_0} \leq C$ ;
2.  $|(f, h)_H| \leq \|T^*h + T_1^*v\|_H$  holds for any  $h \in F, v \in \text{Dom} T_1^* \cap \text{Ker} T_2$ .

To apply lemma 4, we consider, for any fixed  $1 \leq \ell \leq p$ , the following Hilbert spaces and operators.

$$\begin{aligned} H_0 &= \bigwedge^\ell L^2(\Omega, \varphi_1)^{\oplus p}, \quad H_1 = \bigwedge^\ell L_{0,1}^2(\Omega, \varphi_1)^{\oplus p}, \\ H_2 &= \bigwedge^\ell L_{0,2}^2(\Omega, \varphi_1)^{\oplus p}, \quad H = \bigwedge^{\ell-1} L^2(\Omega, \varphi_2)^{\oplus p}, \\ T &= \sqrt{a + \bar{\lambda}} \circ \iota_g, \quad T_1 = \bar{\partial} \circ \sqrt{a + \bar{\lambda}}, \quad T_2 = \sqrt{a} \circ \bar{\partial}. \end{aligned}$$

where these functions  $\varphi_1, \varphi_2, 0 < a, 0 < \lambda \in C^2(\bar{\Omega})$  will be determined later.

Since  $\Omega$  is assumed to be bounded and  $a, \lambda \in C^2(\bar{\Omega})$ , the operator  $T$  is a bounded linear mapping from  $H_0$  to  $H$ .  $T_1, T_2$  are, by definition, both densely defined and closed. The space  $F$  defined by (5) is obviously a closed subspace of  $H$ . From the definition of  $T_1$ , we have

$$\text{Ker} T_1 \subseteq \frac{1}{\sqrt{a + \bar{\lambda}}} \bigwedge^\ell \mathcal{O}(\Omega)^{\oplus p}.$$

Consequently,  $\iota_g^2 = 0$  implies that

$$F \supseteq T(\text{Ker} T_1).$$

It is also easy to see that the adjoint of  $T$  and  $T_1$  are given by

$$T^* = \sqrt{a + \lambda} \iota_g^*, \quad T_1^* = \sqrt{a + \lambda} \bar{\partial}_{\varphi_1}^*$$

where  $\iota_g^*$  is the mapping in (14).



### 3 Division Theorems

First we apply lemma 3 in the simplest situation where the function  $a$  is a constant to establish a division theorem for the Koszul complex. We denote by  $\text{PSH}(\Omega)$  the set of plurisubharmonic functions on  $\Omega$ .

**Theorem 1** Let  $\Omega \subseteq \mathbb{C}^n$  be a pseudoconvex domain,  $g_i \in \mathcal{O}(\Omega)$  ( $1 \leq i \leq p$ ),  $\psi \in \text{PSH}(\Omega)$  and  $\tau > 1$  be a constant. For every  $f \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p}$ , if  $\iota_g f = 0$  and

$$\int_{\Omega} \|f\|^2 \|g\|^{-2(q\ell\tau+1)} e^{-\psi} dV < \infty, \quad (17)$$

then there exists an  $u \in \bigwedge^{\ell} \mathcal{O}(\Omega)^{\oplus p}$  such that

$$\iota_g u = f, \quad \int_{\Omega} \|u\|^2 \|g\|^{-2q\ell\tau} e^{-\psi} dV \leq \frac{\tau}{\tau-1} \int_{\Omega} \|f\|^2 \|g\|^{-2(q\ell\tau+1)} e^{-\psi} dV \quad (18)$$

where  $p \in \mathbb{N}$ ,  $1 \leq \ell \leq p$  and  $q$  is the constant in lemma 1.

Proof. By the standard argument of smooth approximation, the holomorphic extension technique and taking weak limit (proceed as [S72] and [D82]), we can assume without loss of generality that  $\Omega$  is a bounded pseudoconvex domain with smooth boundary,  $g_i \in \mathcal{O}(\Omega) \cap C^\infty(\bar{\Omega})$  ( $1 \leq i \leq p$ ) have no common zeros on  $\bar{\Omega}$  and  $\psi \in \text{PSH}(\Omega) \cap C^\infty(\bar{\Omega})$ .

Given a constant  $\tau > 1$ , we can always find constants  $0 < \lambda < 1 < b$  such that

$$\tau = \frac{b}{1-\lambda}.$$

Set

$$a = 1 - \lambda,$$

then the functions

$$\varphi_1 = q\ell\tau \log \|g\|^2 + \psi, \quad \varphi_2 = (q\ell\tau + 1) \log \|g\|^2 + \psi$$

satisfy the conditions (13) and (15). In this case, we have  $T = \iota_g$  and  $T_1 = \bar{\partial}$ .

Let  $F$  be the closed subspace defined by (5) in section 2 and  $h \in F$ , then we get by lemma 3 that

$$\begin{aligned} |(f, h)_H|^2 &\leq \int_{\Omega} \frac{(b-1)(1-\lambda)}{b} \|h\|^2 e^{-\varphi_2} dV \int_{\Omega} \frac{b}{(b-1)(1-\lambda)} \|f\|^2 e^{-\varphi_2} dV \\ &\leq \int_{\Omega} \frac{b}{(b-1)(1-\lambda)} \|f\|^2 e^{-\varphi_2} dV \cdot \|T^*h + T_1^*v\|_{H_0}^2. \end{aligned}$$

It follows from lemma 4 that there is an  $u_\lambda \in H_0$  such that

$$f = Tu_\lambda = \iota_g u_\lambda$$

and the weighted  $L^2$  norm of  $u$  could be estimated as follows

$$\begin{aligned}
\int_{\Omega} \|u_{\lambda}\|^2 \|g\|^{-2q\ell\tau} e^{-\psi} dV &= \|u_{\lambda}\|_{H_0}^2 \\
&\leq \int_{\Omega} \frac{b}{(b-1)(1-\lambda)} \|f\|^2 e^{-\varphi_2} dV \\
&= \int_{\Omega} \frac{\tau}{(1-\lambda)(\tau+\lambda-1)} \|f\|^2 e^{-\varphi_2} dV \\
&= \int_{\Omega} \frac{\tau}{(1-\lambda)(\tau+\lambda-1)} \|f\|^2 \|g\|^{-2(q\ell\tau+1)} e^{-\psi} dV.
\end{aligned}$$

The desired solution  $u$  follows from the above inequality by taking weak limit of  $u_{\lambda}$  as  $\lambda \rightarrow 0+$ . Q.E.D.

If  $g_1, \dots, g_p$  have no common zeros, then for any  $f \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p}$ , we can use the plurisubharmonic exhaustion function of  $\Omega$  to construct a plurisubharmonic weight function  $\psi$  on  $\Omega$  such that

$$\int_{\Omega} \|f\|^2 \|g\|^{-2(q\ell\tau+1)} e^{-\psi} dV < \infty.$$

Applying theorem 1 on  $\Omega$ , we know there exists some  $u \in \bigwedge^{\ell} \mathcal{O}(\Omega)^{\oplus p}$  such that  $f = \iota_g u$  holds on  $\Omega$ . This gives the following corollary.

**Corollary 1** Let  $\Omega \subseteq \mathbb{C}^n$  be a pseudoconvex domain,  $g_i \in \mathcal{O}(\Omega)$  ( $1 \leq i \leq p$ ). If  $g_1, \dots, g_p$  have no common zeros, then the Koszul complex (1) induces an exact sequence at the level of global sections, i.e. for every  $f \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p}$  ( $1 \leq \ell \leq p$ ) satisfying  $\iota_g f = 0$  there is some  $u \in \bigwedge^{\ell} \mathcal{O}(\Omega)^{\oplus p}$  such that  $f = \iota_g u$ .

**Remark 1** (i) The special case of theorem 1 when  $\ell = 1$  is exactly the celebrated Skoda division theorem ([S72]). If we make use of lemma 2 instead of lemma 3, the proof of theorem 1 will be a little bit easier.

(ii) When the common zero locus of  $g_1, \dots, g_p$  is empty, it is easy to see that the Koszul complex (1) provides a resolution of  $\mathcal{O}_{\Omega} = \bigwedge^p \mathcal{O}_{\Omega}^{\oplus p}$ . Thus corollary 1 also follows from Cartan's theorem B and the De Rham-Weil isomorphism theorem.

To establish division theorems with a nonconstant function  $a$  in (12), we use the technique of Skoda triple which was introduced by Varolin ([V08]). We first recall the definition of a Skoda triple.

**Definition** A Skoda triple  $(\varphi, F, q)$  consists of a positive integer  $q$  and  $C^2$  functions  $\varphi : (1, \infty) \rightarrow \mathbb{R}$ ,  $F : (1, \infty) \rightarrow \mathbb{R}$  such that

$$x + F(x) > 0, [x + F(x)]\varphi'(x) + F'(x) + 1 > 0 \text{ and } [x + F(x)]\varphi''(x) + F''(x) < 0$$

hold for every  $x > 1$ .

It is easy to see that  $(\varepsilon \log x, 0, q)$  is a Skoda triple where  $\varepsilon$  is a positive constant and  $q$  is a positive integer. This example was shown in [V08] where the technique of denominators was also used to construct Skoda triple of the type  $(0, F, q)$ .

Based upon the apriori estimate (16) and lemma 4, the notion of Skoda triple is quite useful to produce examples of division theorems.

**Theorem 2** Let  $\Omega \subseteq \mathbb{C}^n$  be a pseudoconvex domain,  $g_i \in \mathcal{O}(\Omega)$  ( $1 \leq i \leq p$ ),  $\psi \in \text{PSH}(\Omega)$ . We assume that

$$\|g\| < 1 \text{ holds on } \Omega.$$

For every  $f \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p}$ , if  $\iota_g f = 0$  and

$$\int_{\Omega} \|f\|^2 \frac{b}{a(b-1)} \|g\|^{-2(q\ell+1)} e^{\varphi \circ \xi - \psi} dV < \infty, \quad (19)$$

then there exists an  $u \in \bigwedge^{\ell} \mathcal{O}(\Omega)^{\oplus p}$  such that  $\iota_g u = f$  and

$$\int_{\Omega} \|u\|^2 \frac{1}{(a+\lambda)} \|g\|^{-2q\ell} e^{\varphi \circ \xi - \psi} dV \leq \int_{\Omega} \|f\|^2 \frac{b}{a(b-1)} \|g\|^{-2(q\ell+1)} e^{\varphi \circ \xi - \psi} dV \quad (20)$$

where  $p \in \mathbb{N}$ ,  $1 \leq \ell \leq p$ ,  $\xi = 1 - \log \|g\|^2$ ,  $a = \xi + F \circ \xi$ ,  $b = \frac{a\varphi' \circ \xi + F' \circ \xi + 1}{qa\ell} + 1$ ,  $\lambda = \Lambda \circ \xi$ ,  $\Lambda(x) = \frac{-(1+F'(x))^2}{F''(x) + (x+F(x))\varphi''(x)}$ ,  $(\varphi, F, q)$  is a Skoda triple and  $q$  is the constant in lemma 1.

Proof. Given a Skoda triple, we start to construct functions  $\varphi_1, \varphi_2, a > 0, \lambda > 0$  and  $b > 1$  which satisfy conditions (13) and (15).

Set

$$\begin{aligned} \varphi_1 &= -\varphi \circ \xi + \psi + q\ell \log \|g\|^2, \\ \varphi_2 &= -\varphi \circ \xi + \psi + (q\ell + 1) \log \|g\|^2 \end{aligned}$$

then we get

$$\begin{aligned} a\partial_{\alpha}\partial_{\bar{\beta}}\varphi_1 - \partial_{\alpha}\partial_{\bar{\beta}}a - \lambda^{-1}\partial_{\alpha}a\partial_{\bar{\beta}}a &= (a\varphi' \circ \xi + F' \circ \xi + 1 + q\ell a)\partial_{\alpha}\partial_{\bar{\beta}}\log\|g\|^2 \\ &\quad - [a\varphi'' \circ \xi + F'' \circ \xi + \lambda^{-1}(1 + F' \circ \xi)^2]\partial_{\alpha}\xi\partial_{\bar{\beta}}\xi \\ &= (a\varphi' \circ \xi + F' \circ \xi + 1 + q\ell a)\partial_{\alpha}\partial_{\bar{\beta}}\log\|g\|^2. \end{aligned}$$

The last equality follows from the definition of  $\lambda$ .

Now it suffices to choose  $b > 1$  such that

$$a\varphi' \circ \xi + F' \circ \xi + 1 = qa\ell(b-1),$$

i.e.

$$b = \frac{a\varphi' \circ \xi + F' \circ \xi + 1}{qal} + 1.$$

By repeating the argument in the proof of theorem 1, we obtain some  $\tilde{u} \in H_0 = \bigwedge^\ell L^2(\Omega, \varphi_1)^{\oplus p}$  satisfying

$$\bar{\partial}\sqrt{a+\lambda}\tilde{u} = 0, \quad \int_{\Omega} \|\tilde{u}\|^2 \|g\|^{-2q\ell} e^{\varphi \circ \xi - \psi} dV \leq \int_{\Omega} \|f\|^2 \frac{b}{a(b-1)} \|g\|^{-2(q\ell+1)} e^{\varphi \circ \xi - \psi} dV.$$

Thus we get the desired solution  $u = \sqrt{a+\lambda}\tilde{u}$ . Q.E.D.

If we take into account the special Skoda triple  $(\varepsilon \log x, 0, q)$  where  $\varepsilon$  is a positive constant and  $q$  is the constant in lemma 1, applying theorem 2 to this Skoda triple, we have the following corollary.

**Corollary 2** Let  $\Omega \subseteq \mathbb{C}^n$  be a pseudoconvex domain,  $g_i \in \mathcal{O}(\Omega)$  ( $1 \leq i \leq p$ ),  $\psi \in \text{PSH}(\Omega)$ . We assume that

$$\|g\| < 1 \text{ holds on } \Omega.$$

For every  $f \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p}$ , if  $\iota_g f = 0$  and

$$\int_{\Omega} \|f\|^2 \frac{(1 - \log \|g\|^2)^\varepsilon}{\|g\|^{2(q\ell+1)}} e^{-\psi} dV < \infty, \quad (21)$$

then there exists some  $u \in \bigwedge^\ell \mathcal{O}(\Omega)^{\oplus p}$  such that  $\iota_g u = f$  and

$$\int_{\Omega} \|u\|^2 \frac{(1 - \log \|g\|^2)^{\varepsilon-1}}{\|g\|^{2q\ell}} e^{-\psi} dV \leq \frac{q\ell + \varepsilon + 1}{\varepsilon} \int_{\Omega} \|f\|^2 \frac{(1 - \log \|g\|^2)^\varepsilon}{\|g\|^{2(q\ell+1)}} e^{-\psi} dV \quad (22)$$

where  $p \in \mathbb{N}$ ,  $1 \leq \ell \leq p$ ,  $\varepsilon > 0$  is a constant and  $q$  is the constant in lemma 1.

Proof. For the given Skoda triple  $(\varepsilon \log x, 0, q)$ , we have

$$[x + F(x)]\varphi'(x) + F'(x) + 1 = 1 + \varepsilon \text{ and } [x + F(x)]\varphi''(x) + F''(x) = -\frac{\varepsilon}{x}$$

from which it follows that

$$a + \lambda = \frac{(1 + \varepsilon)(1 - \log \|g\|^2)}{\varepsilon}, \quad \frac{b}{a(b-1)} \leq \frac{q\ell + \varepsilon + 1}{\varepsilon + 1}.$$

Hence the desired result follows directly from theorem 2. Q.E.D.

**Remark 2.** Under the assumption that  $\|g\| < 1$  on  $\Omega$ , the integrability condition (21) is obviously weaker than (17).

We know by definition that  $(0, -\frac{1}{2}e^{-\varepsilon(x-1)}, q)$  is another example of Skoda triples where  $\varepsilon$  is a positive constant and  $q$  is the constant in lemma 1. Thus theorem 2 applied to  $(0, -\frac{1}{2}e^{\varepsilon(x-1)}, q)$  gives the following result.

**Corollary 3** Let  $\Omega \subseteq \mathbb{C}^n$  be a pseudoconvex domain,  $g_i \in \mathcal{O}(\Omega)$  ( $1 \leq i \leq p$ ),  $\psi \in \text{PSH}(\Omega)$ . We assume that

$$\|g\| < 1 \text{ holds on } \Omega.$$

For every  $f \in \bigwedge^{\ell-1} \mathcal{O}(\Omega)^{\oplus p}$ , if  $\iota_g f = 0$  and

$$\int_{\Omega} \|f\|^2 \|g\|^{-2(q\ell+1)} e^{-\psi} dV < \infty, \quad (23)$$

then there exists some  $u \in \bigwedge^{\ell} \mathcal{O}(\Omega)^{\oplus p}$  such that  $\iota_g u = f$  and

$$\int_{\Omega} \|u\|^2 \|g\|^{2(-q\ell+\varepsilon)} e^{-\psi} dV \leq C_{\varepsilon} \int_{\Omega} \|f\|^2 \|g\|^{-2(q\ell+1)} e^{-\psi} dV \quad (24)$$

where  $p \in \mathbb{N}$ ,  $1 \leq \ell \leq p$ ,  $\varepsilon$  and  $C_{\varepsilon}$  are both positive constants ( $C_{\varepsilon}$  is determined by  $\varepsilon$ ) and  $q$  is the constant in lemma 1.

Proof. By direct computations, we obtain

$$[x + F(x)]\varphi'(x) + F'(x) + 1 = \frac{\varepsilon}{2}e^{-\varepsilon(x-1)} + 1, \quad [x + F(x)]\varphi''(x) + F''(x) = -\frac{\varepsilon^2}{2}e^{-\varepsilon(x-1)}.$$

Hence we have for  $\xi > 1$

$$\frac{b}{a(b-1)} = \frac{1}{\xi - \frac{1}{2}e^{-\varepsilon(\xi-1)}} + \frac{q\ell}{1 + \frac{\varepsilon}{2}e^{-\varepsilon(\xi-1)}} \leq 2 + q\ell$$

and

$$\begin{aligned} a + \lambda &= \xi - \frac{1}{2}e^{-\varepsilon(\xi-1)} + 2\varepsilon^{-2}(1 + \frac{\varepsilon}{2}e^{-\varepsilon(\xi-1)})^2 e^{\varepsilon(\xi-1)} \\ &\leq [\varepsilon^{-1}e^{\varepsilon-1} + 2(\frac{1}{\varepsilon} + \frac{1}{2})^2] e^{\varepsilon(\xi-1)} \\ &= D_{\varepsilon} e^{\varepsilon(\xi-1)} \end{aligned}$$

where  $D_{\varepsilon}$  is a positive constant determined by  $\varepsilon$ . Now corollary 3 follows from theorem 2 by choosing  $(\varphi, F, q) = (0, -\frac{1}{2}e^{\varepsilon(x-1)}, q)$  and the constant  $C_{\varepsilon}$  in (24) could be taken to be  $(2 + q\ell)D_{\varepsilon}$ . Q.E.D.

**Remark 3.** (i) It is easy to see that when  $\|g\| < 1$  is valid on  $\Omega$  the integrability condition (23) in corollary 3 is weaker than (21) but the estimate (22) for the solution in corollary 2 is stronger than (24). (ii) Comparing corollary 3 with theorem

1, we see that if  $\|g\| < 1$  holds on  $\Omega$  then the constant  $\tau$  in theorem 1 could be chosen to be 1 (the coefficient  $\frac{\tau}{\tau-1}$  on the right hand of (18) should be replaced by  $C_\varepsilon$ ). (iii) It is interesting to compare corollary 3 with the main result of [T00] by setting  $\ell = 1, p = n$  and  $g_i = z_i (1 \leq i \leq n)$ . (iv) We may also choose the Skoda triple more generally to be  $(0, -\eta e^{-\varepsilon(x-1)}, q)$  where  $0 < \eta < 1$  is a constant, but such a choice only results in a different constant  $C_\varepsilon$ . (v) We can use the Skoda triple  $(\varepsilon_1 \log x, -\varepsilon_2 e^{-\varepsilon_3(x-1)}, q)$  to combine the results in corollaries 2 and 3. Here,  $\varepsilon_1 \geq 0, 1 > \varepsilon_2 \geq 0, \varepsilon_3 > 0$  are constants satisfying  $\varepsilon_1 + \varepsilon_2 > 0$ .

**Final Comments.** As mentioned before, one can use the technique of denominators to produce Skoda triples of the type  $(0, F, q)$ . Hence we can deduce from our theorem 2 numerous examples of division theorems. Actually, we can formulate a division theorem for the Koszul complex in the same manner of theorem 2.7 in [V08]. To prove this result, we just need to replace theorem 2.1 in [V08] by our theorem 2 and then repeat its proof.

## 4 Appendix: Proof of Lemma 1

Let  $V, W$  be Hermitian spaces with  $\dim_{\mathbb{C}} V = p, \dim_{\mathbb{C}} W = n$ , and  $\{v_1, \dots, v_p\}, \{w_1, \dots, w_n\}$  be orthonormal bases of  $V, W$  respectively. We denote the dual bases by  $\{v_1^*, \dots, v_p^*\} \subseteq V^*, \{w_1^*, \dots, w_n^*\} \subseteq W^*$ .

Set

$$\begin{aligned} \mathcal{A} &= \sum_{\substack{1 \leq i \leq p \\ 1 \leq \alpha \leq n}} c_{ii_1 \dots i_{\ell-1} \alpha} w_\alpha^* \otimes v_i \in \text{Hom}_{\mathbb{C}}(W, V), \\ \mathcal{B}_1 &= \sum_{\substack{1 \leq i \leq p \\ 1 \leq \alpha \leq n}} b_{i\alpha} v_i^* \otimes w_\alpha \in \text{Hom}_{\mathbb{C}}(V, W), \\ X &= \sum_{1 \leq i \leq p} \bar{a}_i v_i \in V, \quad \theta = \sum_{1 \leq i \leq p} a_i v_i^* \in V^*, \\ \mathcal{B} &= \iota_X(\theta \wedge \mathcal{B}_1) \in \text{Hom}_{\mathbb{C}}(V, W), \end{aligned}$$

then we know by definition the following facts

$$\mathcal{AB} = \iota_X(\theta \wedge \mathcal{AB}_1) \in \text{End}_{\mathbb{C}} V$$

and

$$\text{L.H.S. of (8)} = |\text{Tr} \mathcal{AB}|^2, \quad \text{R.H.S. of (8)} = q \|X\|^2 \|\theta \wedge \mathcal{AB}_1\|^2$$

where both trace and norm are taken with respect to the Hermitian structure on  $V$ . It remains therefore to show

$$|\text{Tr} \mathcal{AB}|^2 \leq q \|X\|^2 \|\theta \wedge \mathcal{AB}_1\|^2.$$

Since the Cauchy-Schwarz inequality gives

$$\|\mathcal{AB}\|^2 \leq \|X\|^2 \|\theta \wedge \mathcal{AB}_1\|^2$$

and

$$|\mathrm{Tr} \mathcal{AB}|^2 \leq \mathrm{Rank}_{\mathbb{C}} \mathcal{AB} \|\mathcal{AB}\|^2,$$

it suffices to estimate the upper bound of  $\mathrm{Rank}_{\mathbb{C}} \mathcal{AB}$ .

Since  $v_{i_1 \dots i_\ell}$  are skew-symmetric on  $i_1, \dots, i_\ell$ , we get

$$\mathrm{Im} \mathcal{AB} \subseteq \mathrm{span}_{\mathbb{C}} \{v_{i_1}, \dots, v_{i_{\ell-1}}\}^\perp.$$

On the other hand, we also have

$$X \in \mathrm{Ker} \mathcal{AB}.$$

We assume, without loss of generality,  $X \neq 0$  then we obtain the following estimate

$$\mathrm{Rank}_{\mathbb{C}} \mathcal{AB} \leq \begin{cases} \min\{p-1, n\}, & \ell = 1; \\ \min\{p-\ell+1, n\}, & \ell \geq 2. \end{cases}$$

This is the desired rank estimate. Q.E.D.

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